KS3$\square$

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There has clearly been much merit and success in extending classical logic to capture certain *modal* notions (e.g., possibility, necessity, knowability, obligation); and, there has similarly been merit and success in extending classical logic to capture certain *many-valued* notions (e.g. indeterminacy, fuzziness). However, there has been relatively little attention paid to the unison of these avenues, namely, *many valued modal logic*.[1] In this paper, I develop one such logic. I focus on a well-known many-valued propositional logic, Kleene Strong Three-Valued Propositional Logic, and I offer its *modal* interpretation, which I will call KS3$^\square$.

My paper has three parts. In the first, I introduce the language of KS3$^\square$: I explain its origin in Kleene's Strong 3-valued Logic, and I motivate for my particular modal interpretation thereof. One may surmise with concern that a three-valued modal theory would somehow *corrupt* its classical components. In the second part of the paper, I show that this worry is unfounded, at least in some important respects. I consider the most influential axioms and systems of modal logic (*D*, *S*, 4, 5, .2, .3), and I demonstrate that models of KS3$^\square$ satisfy those axioms exactly when classical models satisfy them. Lastly, I consider some interesting consequences of KS3$^\square$, and some possible applications.

Understanding Kleene's strong three-valued logic

Kleene's system introduces a third value *U* ('undefined') to accompany the usual bivalent pair of T, F (for truth and falsehood). The nature and status of this third value is a matter of significant philosophical interest in its own right: for example, could *U* mean 'unknown'? Is *U* a *truth*-value, or some other kind of value? Does *U* appear *between* T and F, or *below*, or on some distinct scale altogether? Such questions are highly worthwhile, though I will not dwell on them here beyond the following remarks.

Allow me briefly to be highly exacting about some features of the values in my proposed logic, lest my detractor be inclined to mount this unwelcome argument: 'Suppose φ is undefined at world *w*. Well, then, since it is undefined,

[1] Fitting, by contrast, has contributed a significant proportion of the work on this area. M. Fitting, "Many-valued modal logics," *Fundam. Inform.* 15, no. 3-4 (1991).

it is not true. But that which is not true is *false*. So, contrary to what we assumed, φ is false at *w*.' This argument trades on the ambiguity of the notion of 'not true,' an ambiguity which must be dissolved before we proceed. So: I assume that 'T' and 'F' stand for 'true' and 'false', which is to say that they are *truth*-values, and they have the meaning we typically assign them in classical logic (and philosophy, and life in general). Consequently, T and F are non-identical—as much so as 0 is non-identical to 1—and are quite contrary to each other. Importantly, *the negation of the one implies the truth of the other*. I maintain—as Kleene does—that 'U' stands for 'undefined.' This third value, I suggest, is not a third *truth*-value in the same sense (or on the same scale) as T and F; if anything, U indicates the *absence* of a truth-value. At the very least, U does not admit of comparison with T and F in the way that the latter pair do. As Kleene puts it, "*U* is thus not on a par with the other two *T* and *F* in our theory."[2]

This suggests that the logic I am developing is not to be understood as a 'logic of vagueness,' or a paraconsistent logic, since nothing I have said would require any departure from an entirely classical understanding of truth—and to the best of my apprehension, Kleene had no such intention with his three-valued logic. That is not to say that my logic could not be put to fruitful ends in theorizing about vagueness or indeterminacy, only that it is not mandatory.

I interpret and develop Kleene's three-valued logic in the following way. First, as mentioned, only T and F are truth-values in the ordinary sense, and they behave altogether classically. U, therefore, is a distinct kind of value; I am motivated by Kleene's discussion of *algorithms* in its interpretation. Consider the computations of a Turing Machine, proceeding from state to state, evaluating some formula φ in propositional logic and returning a value in {T,F,U}. Suppose that the program on the Turing tape in question instructs the machine to compute the value of some very long conjunctive sentence φ ($\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \ldots \wedge \varphi_n$). It seems quite intuitive to me that in the start state φ's value may commence at U; the machine would then 'hunt' for an F, at which point the whole conjunction would be falsified, and the machine would return an F as output, and halt.

I think this gives us good grounds to employ a third value U in our theorizing, but the foregoing example may make it seem as though U is just a *placeholder*

[2] Stephen Cole Kleene, *Introduction to metamathematics* (Amsterdam: North Holland Publising Co., 1952). 333.

for T or F; that any φ which evaluates to U is really a classical value in disguise, or it certainly will be a classical value at some later stage, or that it is simply *unknown for now* which classical value is 'behind' that U. (It seems Haack (1974) may think something like this.) But I think this is to shortchange Kleene's vision, and considerations of Turing machines are instructive. Consider the behavior of such machines in computing *partial* functions, which may perhaps be likened to the following case. Consider the Turing machine program for $\vee$, here computing the overall value of a long disjunctive sentence φ ($\varphi_1 \vee \varphi_2 \vee \varphi_3 \vee \ldots \vee \varphi_n$). Then suppose I put in a tape with an infinite number of U's—say, one U for each of the natural numbers. And, exploiting compactness and non-standard models, I place a T at the very top of the structure. We can imagine the Turing machine whirring endlessly from U to U without limit, hunting for a T, but all the while affirming that the value of the sentence is U since it had not reached the end of the tape. But, *ex hypothesi*, it never will reach the end of the tape.

I think this gives us a stronger sense of 'undefined.' And, since we already have a strong sense of truth and falsehood, let's consider the finer details of KS3$^{\square}$. I construct KS3$^{\square}$ with the following Kleene/Kripke-inspired question kept firmly in mind: *in what circumstances is φ determinately true or false?* I then fix the answers to those questions, as far as possible, and the remaining values are undefined.

In this way, Kleene logic may be easily constructed provided we have a grip on what makes classical formulae true or false. For example, we know that a conjunction ($\varphi \wedge \psi$) is true where *both* of its conjuncts are true, and false where *either* conjunct is false. The 'or' connective, in the alternative, is quite *permissive*: it is true where *any* disjunct is true, and false only where *all* disjuncts are false. Kleene's system sustains these basic intuitions, and fixes any remaining cases as U. The truth tables make this clear.[3]

[3] M. Bergmann, *An introduction to many-valued and fuzzy logic: semantics, algebras, and derivation systems* (Cambridge Univ Pr, 2008). 71-75

The Kleene connectives are defined according to their truth tables as follows:

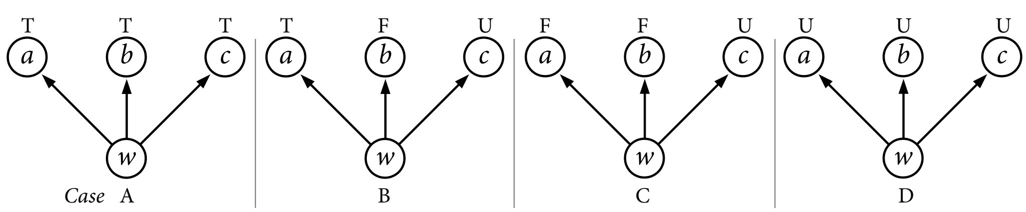
P	**Q**	**¬P**	**P v Q**	**P & Q**	**P→Q**	**P↔Q**
T	**T**	F	T	T	T	T
T	**U**	F	T	U	U	U
T	**F**	F	T	F	F	F
U	**T**	U	T	U	T	U
U	**U**	U	U	U	U	U
U	**F**	U	U	F	U	U
F	**T**	T	T	F	T	F
F	**U**	T	U	F	T	U
F	**F**	T	F	F	T	T

The foregoing Kleene transformation proceeded by finding that *feature* which made true propositions true, and extending it to the three-valued domain. Similarly, I believe the Kleene system recommends a fairly natural way to treat the modal operators. Broadly construed, modal logic permits us to capture notions of *possibility* and *necessity* in a formal system; this is achieved by way of the introduction of two operators, ◇ and □. These operators are understood in their usual way as follows:

□φ: "It is necessary that φ" or "necessarily φ" or "φ obtains with necessity."

◇φ: "It is possible that φ" or "possibly φ" or "φ is a possibility."

In classical modal logic, we held that necessity requires truth at *all* possible worlds; and possibility requires truth at *some* possible world. Holding onto this intuition, consider the following models:



In the models, world *w* accesses worlds *a, b,* and *c*. The value above each of the worlds indicates the truth of a proposition φ at that world. So, in case A, we have a paradigm example of necessity: □φ is true at *w*, since φ is true at all worlds accessible to *w*. However, clearly this is not true in case B, since φ is false at some world there, and φ is U at some other world. But note: even if *c* were not U, it would not matter to us, since □φ had already been invalidated by the presence of an F. However, it seems correct to say that ◇φ is true at *w* in case B, since there is some world *a* accessible to *w* at which φ is true; and, again, this is not threatened by the presence of the U in the universe. Finally, turning to case D, we have a paradigm instance of undefined-ness, if there is such a thing! Clearly, both □φ and ◇φ at *w* will evaluate to U.

The language of KS3$^{\Box}$

Now that we are intuitively satisfied, let us regiment this as KS3$^{\Box}$ formally. We already know what it is for a formula of propositional logic to be true or false: this is given by its truth function. However, with respect to modality, we are not merely interested in whether a formula is true, but whether it is *necessarily* or *possibly* true. This requires additional apparatus: I will employ a possible-worlds analysis of modality, and I will express these conventionally using Kripke frames.[4]

Primitive vocabulary

Propositional variables: P, Q, R, P1, P2, P3, Pn etc.

Connectives: ¬, ∧, ∨, ↔, →

Modal operators: □, ◇

Parentheses: (,)

Well formed formulae

Any propositional variable is a well formed formula.

If φ and ψ are well formed then so are:

$\neg\varphi \mid \varphi\wedge\psi \mid \varphi\vee\psi \mid \varphi\rightarrow\psi \mid \varphi\leftrightarrow\psi \mid \Box\phi \mid \Diamond\phi \mid (\phi)$

And nothing else is a wff.

Possible Worlds, Models, and Kripke Frames

The structure of a possible worlds model is determined by its *frame* ⟨*W*, *R*⟩, which specifies the objects in the universe (*W*) and how they are related (*R*). To this it an *interpretation* V_w is added; and so we have the modal model, a triple $\mathcal{M}\langle W,R,V_w\rangle$, the parts of which are:

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[4] I found van Benthem's J. Van Benthem, *Modal logic for open minds* (CSLI publications, 2010), new book very helpful with these constructions.

W: a non-empty set of possible worlds (or objects, or nodes, or whatever, but I will consistently refer to the items as worlds). For example, $W = \{w_1, w_2, w_3, w_4\}$.
R: a binary *directed accessibility relation* between worlds in W. For example, Ruv means that world u can access world v; equivalently, v is accessible from u.
V_w: an interpretation function which assigns values in {T,F,U} to proposition letters P at world w. For example, $V_w(\mathrm{P}) = \mathrm{T}$.

Consider a model $\mathscr{M}\langle W,R,V_w\rangle$ as above. We say that a proposition φ is *true at world w if and only if* $V_w(\varphi) = 1$. Intuitively, the function $V_w(\varphi)$ 'looks' at world w and 'sees' whether φ is true at that world; if it is, it returns the value T, and if it is false it returns F, and U otherwise. From here, we define the notion of a formula φ being *satisfied* (or *true)* in $\mathscr{M}$ at world $w \in W$ as follows:

$$\mathscr{M}_w \vDash \varphi \begin{cases} T, & V_w(\varphi) = T; \text{that is}, \varphi \in w \\ F, & V_w(\varphi) = F; \text{that is}, \varphi \notin w \\ U, & otherwise \end{cases}$$

$$\mathscr{M}_w \vDash \neg\varphi \begin{cases} T, & V_w(\varphi) = F; \text{that is}, \varphi \notin w \\ F, & V_w(\varphi) = T; \text{that is}, \varphi \in w \\ U, & otherwise \end{cases}$$

$$\mathscr{M}_w \vDash \varphi \vee \psi \begin{cases} T, & V_w(\varphi) = T \text{ or } V_w(\psi) = T \\ F, & V_w(\varphi) = F \text{ and } V_w(\psi) = F \\ U, & otherwise \end{cases}$$

$$\mathscr{M}_w \vDash \varphi \wedge \psi \begin{cases} T, & V_w(\varphi) = T \text{ and } V_w(\psi) = T \\ F, & V_w(\varphi) = F \text{ or } V_w(\psi) = F \\ U, & otherwise \end{cases}$$

$$\mathscr{M}_w \vDash \varphi \rightarrow \psi \begin{cases} T, & V_w(\varphi) = F \text{ or } V_w(\psi) = T \\ F, & V_w(\varphi) = T \text{ and } V_w(\psi) = F \\ U, & otherwise \end{cases}$$

$$\mathscr{M}_w \vDash \forall x\varphi(x) \begin{cases} T, & V_w(\varphi(\bar{a}) = T \text{ for every } a \text{ in } W \\ F, & V_w(\varphi(\bar{a}) = F \text{ for some } a \text{ in } W \\ U, & otherwise \end{cases}$$

$$\mathcal{M}_w \vDash \exists x\varphi(x) \begin{cases} T, & V_w(\varphi(\bar{a}) = T \text{ for some } a \text{ in } W \\ F, & V_w(\varphi(\bar{a}) = F \text{ for all } a \text{ in } W \\ U, & otherwise^5 \end{cases}$$

Lastly, we introduce the modal operators. By ◇φ we intend that φ is true at *some possible world* among the set of worlds in *W*, and *accessible* from the position in question. More precisely, ◇φ is true in a model $\mathcal{M}$ at world *w* just if there exists some world *v* accessible from world *w* (*Rwv*), and φ is true at *v*, that is, $V_v(\varphi) = 1$. Correspondingly, ◇φ is false in a model $\mathcal{M}$ at world *w* just if there are *no worlds v* at which φ is true. So, formally:

$$\mathcal{M}_w \vDash \Box\varphi \begin{cases} T, & \forall v(Rwv \to \mathcal{M}_v \vDash \varphi) \\ F, & \exists v(Rwv \wedge \mathcal{M}_v \vDash \neg\varphi) \\ U, & otherwise \end{cases}$$

$$\mathcal{M}_w \vDash \Diamond\varphi \begin{cases} T, & \exists v(Rwv \wedge \mathcal{M}_v \vDash \varphi) \\ F, & \forall v(Rwv \to \mathcal{M}_v \vDash \neg\varphi) \\ U, & otherwise \end{cases}$$

Note immediately that certain crucial relations are sustained. In particular, $V_w(\varphi)= V_w(\neg(\neg\varphi))$. Also, in the modal case, possibility and necessity are classical dualities, and this is preserved: $\neg\Diamond\varphi \leftrightarrow \Box\neg\varphi$. Lastly note that it would be simple to prove the distribution axioms: $\Box(\varphi\to\psi)\to(\Box\varphi\to\Box\psi)$; and $\Diamond(\varphi\vee\psi)\leftrightarrow(\Diamond\varphi\vee\Diamond\psi)$.

Since the operators have been defined recursively, we know what it is for a complex sentence, say, $\Box(P \vee Q)$ to hold in a given model $\mathcal{M}$ at world *k*. We simply consider the set of *k*-accessible worlds $v \subseteq W$ such that *Rkv*. Then, looking at those accessible worlds, we ask, is the formula φ true? In this case, we are asking if it is true that $\mathcal{M}_k \vDash \Box(P \vee Q)$. This sentence will be true just if at every *v* such that *Rkv*, either φ is true, or ψ is true (or both). It will be false just in case there is some possible world in which *neither* φ nor ψ is true.

[5] I found Blamey's (Stephen Blamey, "Partial logic," in *Handbook of Philosophical Logic*, ed. Dov M. Gabbay and F. Guenthner (Amsterdam: Kluwer Academic Publishers, 2002), 263) recommendations concerning this formulation of the quantifiers very helpful, as was his chapter in general.

Verifying the axioms

Let's consider now how KS3$^{\square}$ fares in satisfying the axioms of classical modal logic, those formulae which give structure to the systems (e.g., *K*, S4, S5, etc.) of modal logic. In classical modal logic, the axioms are satisfied provided the frame has the appropriate relational structure; for example, Kripke models on any frame *F* satisfy axiom *5* if and only if *F* is symmetrical. I consider six important axioms (*D, S, 4, 5, .2,* and *.3*) and I demonstrate that these axioms have precisely the same *satisfaction conditions* in KS3$^{\square}$ as they do in classical modal logic.

Each of these proofs has two parts, since the claims are *biconditional*; for example, a frame F satisfies axiom D *if and only if* F is serial. My method of proof is the same throughout the six axioms, so I explain it once at the outset. Starting with the proof from left to right, I show that a frame satisfies the axiom in question *only if it has a specific relational structure* (e.g., is reflexive, transitive, serial, etc.). To prove this, I argue by *contraposition*, and suppose that the frame *lacks* the relational structure in question. I then construct (and illustrate) a model which is an arbitrary instance of this frame, and show that the model does not satisfy the axiom. In the proof from right to left, I show that any Kripke frame with the appropriate relational structure will satisfy the axiom in question. I argue directly from the formal specification of the relation (e.g., reflexive: *Rxx*) to the truth of the desired axiom.

My illustrations of the models follow convention in using circles to represent worlds and arrows to represent access between worlds in the direction of the arrow. A formula next to a circle indicates that it is true at that world. While I use black unbroken arrows to show which worlds access each other in the model, I sometimes use dashed turquoise arrows to show which relations are *missing* from the model, that is, which connections would be required to satisfy the axiom.

***D*:** $\Box\varphi \rightarrow \Diamond\varphi$

Claim: Kripke models on frame F satisfy D if and only if F is serial (every world accesses some world or other).

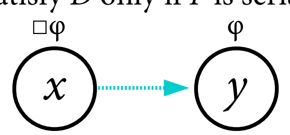
Proof:

a) Kripke models on frame F satisfy D only if F is serial.

Contraposition. Suppose that the accessibility relation R in frame $F^*\langle W,R\rangle$ is *not* serial:

$$\neg(\forall x \exists y(Rxy)\text{, or equivalently, } \exists x \forall y(\neg Rxy);\ x,y \in W.$$

Since F^* is not serial, there is some world w which accesses no worlds—not even itself. Now, let us construct a modal model $\mathscr{M}\langle W,R,V_w\rangle$ on frame F^* and demonstrate that it does not satisfy D. I illustrate this model below, in which $\Box\varphi$ is true at world x and φ is true at world y, but x does not access y (as would suffice for D, see turquoise arrow). The accessibility relation R is *empty* since no world accesses any other; consequently, $\mathscr{M}_x \vDash \Box\varphi$, defined as $\forall y(Rxy \rightarrow \mathscr{M}_y \vDash \varphi)$, is *vacuously* true. Now, if axiom D were satisfied, then it would follow that $\mathscr{M}_x \vDash \Diamond\varphi$. However, $\mathscr{M}_x \vDash \Diamond\varphi$ is false since there is no x-accessible world at which φ is true. Therefore, F^* does not model D, and so Kripke models on frame F satisfy D only if F is serial.



b) Kripke models on frame F satisfy D if F is serial.

Consider modal model $\mathscr{M}\langle W,R,V_w\rangle$ where the accessibility relation R is serial; that is,

$$\forall x \exists y(Rxy);\ x,y \in W.$$

Consider any arbitrary world w, and, since R is serial, note that there is some world v accessible from w. Suppose φ is true at all worlds accessible to w; that is $\mathscr{M}_w \vDash \Box\varphi$. Then, by definition $\forall y(Rwy \Rightarrow \mathscr{M}_y \vDash \varphi)$, so φ is true at v. But then there is some possible world v accessible to w where φ is true at v. That is, $\mathscr{M}_w \vDash \Diamond\varphi$. So, by conditional proof, $\mathscr{M}_w \vDash \Box\varphi \rightarrow \Diamond\varphi$. Since w was an arbitrary world in $\mathscr{M}$ and there are no further assumptions, we will drop the subscripted 'w' and say $\mathscr{M} \vDash \Box\varphi \rightarrow \Diamond\varphi$; that is, D. So Kripke models on frame F satisfy D if F is serial.

Therefore, Kripke models on frame F satisfy D if and only if F is serial.■

S: $\Box\varphi \rightarrow \varphi$

Claim: Kripke models on frame F satisfy S if and only if F is reflexive (every world should be self-accessing).

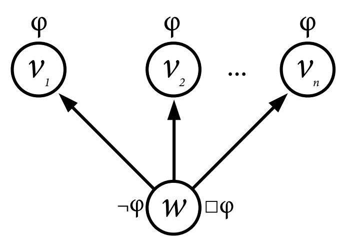
Proof:

a) Kripke models on frame F satisfy S only if F is reflexive.

Contraposition. Suppose that the accessibility relation R in frame $F^*\langle W,R\rangle$ is *not* reflexive:

$\neg\forall x(Rxx)$, or equivalently, $\exists x(\neg Rxx)$; $x \in W$.

Consider modal model $\mathscr{M}\langle W,R,V_w\rangle$ on F* as illustrated below, containing arbitrary world w which does not access itself. Consistent with this model, suppose $\mathscr{M}_w \vDash \Box\varphi$, so φ is true at all worlds v accessible to w, and suppose also that φ is determinately *false* at w itself: $\mathscr{M}_w \vDash \neg\varphi$, and so $\neg(\mathscr{M}_w \vDash \neg\varphi)$. If S were satisfied, then, since $\mathscr{M}_w \vDash \Box\varphi$, it would follow that $\mathscr{M}_w \vDash \varphi$. But since this implication does not obtain, the model does not satisfy S. So Kripke models on frame F satisfy S only if F is reflexive. Note that the model holds for *all* worlds v, and is consistent with $v = \{v_1, v_2, v_3, \dots, v_n\}$ or, of course, $v = \{\emptyset\}$.



b) Kripke models on frame F satisfy S if F is reflexive.

Consider modal model $\mathscr{M}\langle W,R,V_w\rangle$ where the accessibility relation R is reflexive; that is,

$\forall w(Rww)$; $w \in W$.

Consider any arbitrary world w. Assume that $\mathscr{M}_w \vDash \Box\varphi$, so φ is determinately true at all worlds v accessible from w; that is, $\forall v(Rwv \Rightarrow \mathscr{M}_v \vDash \varphi)$. Since R is reflexive, w itself is accessible from w: Rww. Consequently, φ must be true at w also: $\mathscr{M}_w \vDash \varphi$. So, by conditional proof from our assumption that $\Box\varphi$, we derive: $\mathscr{M} \vDash \Box\varphi \rightarrow \varphi$; that is, axiom S. So Kripke models on frame F satisfy S if F is reflexive.

Therefore, Kripke models on frame F satisfy S if and only if F is reflexive.■

4. $\Box\varphi\rightarrow\Box\Box\varphi$

Claim: Kripke models on frame F satisfy *4* if and only if F is transitive.

Proof:

a) Kripke models on frame F satisfy 4 only if F is transitive.

Contraposition. Suppose that the accessibility relation R in frame $F^*\langle W,R\rangle$ is *not* transitive:

$\neg\forall x\forall y\forall z((Rxy \wedge Ryz)\rightarrow Rxz)$. Equivalently, we assert that: $\exists x\exists y\exists z((Rxy\wedge Ryz)\wedge\neg Rxz)$, $x,y,z \in W$ (and note that this equivalence is preserved by the definitions of KS3$^{\Box}$).

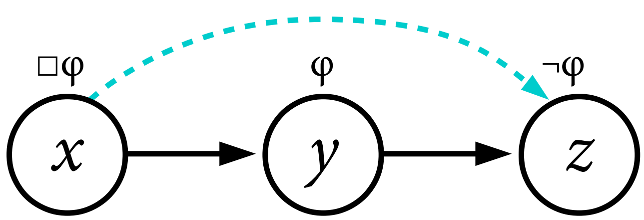
Consider modal model $\mathscr{M}\langle W,R,V_w\rangle$ on F^* as illustrated below, containing arbitrary worlds x, y, and z, in which we fix the following facts:

(1) As the frame requires, $\exists x\exists y\exists z((Rxy\wedge Ryz)\wedge\neg Rxz)$. So, in our example, world z is accessible to y, and y is accessible to x, but z is not accessible to x.

(2) $\mathscr{M}_x\vDash\Box\varphi$; that is, φ is true at all worlds y accessible to x.

(3) $\mathscr{M}_z\vDash\neg\varphi$; that is, at world z, $\neg\varphi$ is true.

Now, if axiom *4* were satisfied in $\mathscr{M}$, then, since $\mathscr{M}_x\vDash\Box\varphi$, it would follow that $\mathscr{M}_x\vDash\Box\Box\varphi$ (by which we should understand: 'at all x-accessible worlds, y, it is true that at all y-accessible worlds, z, it is true that φ'). But φ is determinately false at z, so the implication does not obtain. So, F^* does not satisfy *4*, and thus Kripke models on frame F satisfy *4* only if F is transitive. (See the turquoise arrows indicating this: if the model were transitive, then Rxz, and since $\mathscr{M}_x\vDash$ $\Box\varphi$, it would follow that $\mathscr{M}_z\vDash\varphi$ and $\mathscr{M}_x\vDash\Box\Box\varphi$, and *4* would be satisfied.)



b) Kripke models on frame F satisfy 4 if F is transitive.

Consider modal model $\mathscr{M}\langle W,R,V_w\rangle$ where the accessibility relation R is transitive; that is,

$\forall x\forall y\forall z((Rxy \wedge Ryz)\rightarrow Rxz)$; $x,y,z \in W$.

Assume that it is true that $\mathscr{M}_x \vDash \Box\varphi$, which implies that $\forall y(Rxy \rightarrow \mathscr{M}_y \vDash \varphi)$, so at any world y such that Rxy, it must be true that φ. Then, consider any world z such that Ryz. Since R is transitive, then, since Rxy and Ryz, it follows that Rxz. Since $\mathscr{M}_x \vDash \Box\varphi$, φ is true at all worlds accessible to x; and so φ must be true at z also: $\mathscr{M}_z \vDash \varphi$. From the vantage point of y, it is true that $\mathscr{M}_y \vDash \Box\varphi$. So, by conditional proof from our original assumption, $\mathscr{M}_x \vDash \Box\varphi \rightarrow \forall y(Rxy \rightarrow \mathscr{M}_y \vDash \Box\varphi)$. So S is satisfied, and thus Kripke models on frame F satisfy S if F is transitive.

By (a) and (b) Kripke models on frame F satisfy S if and only if F is transitive.■

5. $\Diamond\Box\varphi \rightarrow \varphi$

Claim: Kripke models on frame *F* satisfy *5* if and only if *F* is symmetrical.

Proof:

a) Kripke models on frame F satisfy 5 only if F is symmetrical.

Suppose that the accessibility relation R in frame $F^*\langle W,R\rangle$ is *not* symmetrical, that is:

$$\neg(\forall x \forall y(Rxy \rightarrow Ryx)), \text{ or, equivalently, } \exists x \exists y(Rxy \wedge \neg Ryx);\ x,y \in W.$$

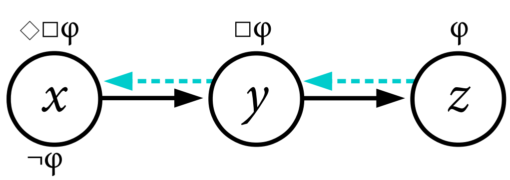
Consider modal model $\mathscr{M}\langle W,R,V_w\rangle$ on F* as illustrated below, containing arbitrary worlds *x*, *y* and *z*. Specifically, we fix the following facts in the model $\mathscr{M}$:

(1) world *x* accesses world *y*, but *y* does not symmetrically access *x*;

(2) φ is determinately *false* at *x*, so $\neg(\mathscr{M}_x \vDash \varphi)$;

(3) $\Box\varphi$ is true at *y*, so, for all worlds *z* such that *z* is accessible from *y*, φ is true at *z*.

So, from the 'vantage point' of *x*, there is a possible world *y* at which $\Box\varphi$, so $\mathscr{M}_x \vDash \Diamond\Box\varphi$. Now, if $\mathscr{M}$ satisfied *5*, then φ would be true at *x*; that is, $\mathscr{M}_x \vDash \varphi$. But as specified $\neg(\mathscr{M}_x \vDash \varphi)$, thus $\mathscr{M}$ does not satisfy *5*. Therefore, Kripke models on frame *F* satisfy *5* only if *F* is symmetrical. (See the turquoise arrows indicating this: if *Ryx*, then, since $\mathscr{M}_y \vDash \Box\varphi$, it would follow that $\mathscr{M}_x \vDash \varphi$, and *5* would be satisfied.)



b) Kripke models on frame F satisfy 5 if F is symmetrical.

Consider modal model $\mathcal{M}\langle W,R,V_w\rangle$ in which the accessibility relation R is symmetrical; that is,

$$\forall x\forall y(Rxy\rightarrow Ryx);\ x,y\in W$$

Consider any world w at which it is true that $\mathcal{M}_w\vDash\Diamond\Box\varphi$. Then, for some world u such that Rwu, $\mathcal{M}_u\vDash\Box\varphi$; that is, for *all* worlds v such that Ruv, $\mathcal{M}_v\vDash\varphi$. Since R is symmetric, if u is accessible from w, then w is accessible from u also. And, since φ is true at all worlds accessible from u, φ is true at w also: that is, $\mathcal{M}_w\vDash\varphi$. So, by conditional proof, $\mathcal{M}_w\vDash\Diamond\Box\varphi\rightarrow\varphi$, that is, *5*. So, Kripke models on frame F satisfy *5* if F is symmetrical.

By (a) and (b) Kripke models on frame F satisfy 5 if and only if F is symmetrical.■

.2 **$\Diamond\Box\varphi\rightarrow\Box\Diamond\varphi$.** *Also known as axiom C (Chellas) or H (Hinktikka)*

Claim: Kripke models on frame F satisfy *.2* if and only if F is convergent.

Proof:

a) Kripke models on frame F satisfy .2 only if F is convergent.

Contraposition. Suppose that the accessibility relation R in frame $F^*\langle W,R\rangle$ is *not* convergent:

$$\neg(\forall w\forall x\forall y((Rwx\wedge Rwy)\rightarrow\exists z(Rxz\wedge Ryz)));\ \text{i.e.,}$$

$$\exists w\exists x\exists y((Rwx\wedge Rwy)\wedge\neg\exists z(Rxz\wedge Ryz))\ w,x,y,z\in W.$$

Consider modal model $\mathcal{M}\langle W,R,V_w\rangle$ on F^*, containing arbitrary worlds w and x. Specifically, we fix the following facts in the model $\mathcal{M}$:

(1) $\mathcal{M}_w\vDash\Diamond\Box\varphi$; there is some world x accessible from w (i.e., Rwx), such that…

(2) …$\mathcal{M}_x\vDash\Box\varphi$; So, for all worlds z such that Rxz, φ is true at z.

In $\mathcal{M}$, world x accesses no worlds at all, so (2) is vacuously true. If *.2* were satisfied then it would follow that $\mathcal{M}_w\vDash\Box\Diamond\varphi$. However, there is no x-accessible world y at which φ is true. So, *.2* is not satisfied and thus Kripke models on frame F satisfy *.2* only if F is directed.

b) Kripke models on frame F satisfy .2 if F is convergent.

Consider modal model $\mathcal{M}\langle W,R,V_w\rangle$ in which the accessibility relation R is convergent; that is,

$\forall w\forall x\forall y((Rwx\wedge Rwy)\rightarrow\exists z(Rxz\wedge Ryz); w,x,y,z\in W.$

Consider arbitrary world w, and assume that $\mathcal{M}_w\vDash\Diamond\Box\varphi$. Then, there exists some world x such that Rwx, $\mathcal{M}_x\vDash\Box\varphi$; and so, for *all* worlds z such that Rxz, $\mathcal{M}_z\vDash\varphi$. This much was true in the counter-model above; here, however, since R is convergent, there must *really exist* some world z such that Rxz and, since φ is true at all z such that Rxz, $\mathcal{M}_z\vDash\varphi$. (Because R is convergent, it is not possible for $\mathcal{M}_x\vDash\Box\varphi$ to be vacuously satisfied.) Now, by our construction it is automatically true that $\mathcal{M}_w\vDash\Box\Diamond\varphi$. So, by conditional proof from our original assumption,

$\mathcal{M}_w\vDash\Diamond\Box\varphi\rightarrow\mathcal{M}_w\vDash\Box\Diamond\varphi$; that is, *.2* is satisfied in $\mathcal{M}$, and so Kripke models on frame F satisfy *.2* if F is directed.

By (a) and (b), Kripke models on frame F satisfy .2 if and only if F is convergent. ■

.3 $(\Diamond\varphi\wedge\Diamond\psi)\rightarrow[\Diamond(\phi\wedge\Diamond\psi)\vee\Diamond(\phi\wedge\psi)\vee\Diamond(\Diamond\varphi\wedge\psi)]$

Claim: Kripke models on frame F satisfy *.3* if and only if F is non-branching.

Proof:

a) Kripke models on frame F satisfy .3 only if F is non-branching.

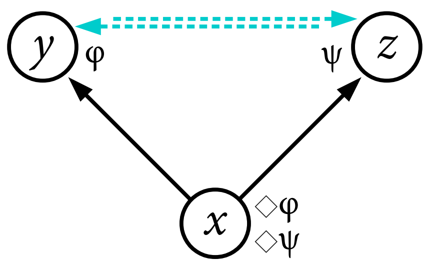
Contraposition. Suppose that the accessibility relation R in frame $F^*\langle W,R\rangle$ is *not* non-branching:

$\neg(\forall x\forall y\forall z((Rxy\wedge Rxz)\rightarrow(Ryz \vee Rzy \vee y=z)))$; i.e., $\exists x\exists y\exists z((Rxy\wedge Rxz\wedge\neg Ryz\wedge\neg Rzy\wedge y\neq z); x,y,z\in W.$

Consider modal model $\mathcal{M}\langle W,R,V_w\rangle$ on F* as illustrated below, containing arbitrary worlds x,y and z. Specifically, we fix the following facts in the model $\mathcal{M}$:

(1) Assume $\mathcal{M}_x\vDash\Diamond\varphi\wedge\Diamond\psi$. This is satisfied where $\mathcal{M}_x\vDash\Diamond\varphi$ and $\mathcal{M}_x\vDash\Diamond\psi$. So:
(2) $\mathcal{M}_y\vDash\varphi$
(3) $\mathcal{M}_z\vDash\psi$
(4) Consonant with the requirements for F^*, $\neg Ryz$ and $\neg Rzy$ and $y\neq z$.

Now, if axiom *.3* were satisfied (see turquoise arrows), then, since $\mathscr{M}_x \vDash \Diamond\varphi \wedge \Diamond\psi$, it would follow that $\mathscr{M}_x \vDash \Diamond(\varphi \wedge \Diamond\psi) \vee \Diamond(\varphi \wedge \psi) \vee \Diamond(\Diamond\varphi \wedge \psi)$. However, all of these disjuncts are false in $\mathscr{M}$. Thus, *.3* is not satisfied, and Kripke models on frame *F* satisfy *.3* only if *F* is non-branching



b) Kripke models on frame F satisfy .3 if F is non-branching

Consider modal model $\mathscr{M}\langle W,R,V_w\rangle$ in which the accessibility relation *R* is non-branching:

$$\forall x \forall y \forall z((Rxy \wedge Rxz) \rightarrow (Ryz \vee Rzy \vee y{=}z);\ x,y,z \in W.$$

Consider modal model $\mathscr{M}\langle W,R,V_w\rangle$ on F* as illustrated below, containing worlds *x, y* and *z*, and consider any world *x* such that $\mathscr{M}_x \vDash \Diamond\varphi \wedge \Diamond\psi$. This is satisfied where $\mathscr{M}_x \vDash \Diamond\varphi$ and $\mathscr{M}_x \vDash \Diamond\psi$. So: $\mathscr{M}_y \vDash \varphi$ and $\mathscr{M}_z \vDash \psi$ (and we leave it open for now whether $y{=}z$). Since *F* is non-branching, and since $Rxy \wedge Rxz$, it follows that (i) *Ryz* or (ii) *Rzy* or (iii) $y{=}z$. Suppose

i. *Ryz*. Then it will be true that $\mathscr{M}_y \vDash (\varphi \wedge \Diamond\psi)$, and so $\mathscr{M}_x \vDash \Diamond(\varphi \wedge \Diamond\psi)$ So *.3* is satisfied.

ii. *Rzy*. Then it will be true that $\mathscr{M}_z \vDash (\Diamond\varphi \wedge \psi)$ so $\mathscr{M}_x \vDash \Diamond(\Diamond\varphi \wedge \psi)$ So *.3* is satisfied.

iii. $y{=}z$. Then it will be true that $\mathscr{M}_y {=} \mathscr{M}_z \vDash (\varphi \wedge \psi)$ so $\mathscr{M}_x \vDash \Diamond(\varphi \wedge \psi)$ So *.3* is satisfied.

So, *.3* is satisfied, and thus Kripke models on frame *F* satisfy *.3* if *F* is non-branching.

By (a) and (b), Kripke models on frame F satisfy .3 if and only if F is non-branching.■

Some Features of KS3$^{\Box}$

So, the traditional axioms of modal logic are satisfied in KS3$^{\Box}$ under their normal (classical) frame constraints; we did not need to amend the frame conditions at all in order for these validities to emerge. This result is quite interesting—and possibly puzzling: since KS3$^{\Box}$ has an additional value, U, how is it that this feature has failed to disrupt the classical configuration? The answer is simple, in some sense: it is that in all the axioms with which we are concerned, the U value *never appears*.

Consider the axioms with which we are concerned, and note that by our definitions for KS3$^{\Box}$ whenever it is true that $\mathcal{M}_x \vDash \Box\varphi$, this means that for all worlds *y* such that *Rxy*, φ is *determinately true* at *y*, and similarly for $\Diamond\varphi$. And, on those occasions where $\mathcal{M}_x \vDash \neg\varphi$, this means that φ is *determinately false* at *w*. Importantly, the duality of T and F and φ and ¬¬φ is preserved, each being the negation of the other. Thus, since we restricted our attention to classically defined cases, we had no reason to worry that U would enter the picture at all. As Kripke noted, Kleene's logic is a highly useful logic for precisely this reason; it is able to permit certain new values to emerge, while preserving any values which have already been fixed by their classical components.[6] The U's were not able to 'infect' any of their classical counterparts.

It may be suggested that this result was *deceptively* simple, however, because the introduction of modal operators into a system may obscure certain differences[7]; in any event, there are other circumstances in which the peculiarities of KS3$^{\Box}$ are more vivid. In particular, consider that it is a commonplace of Kleene Strong propositional logic that there are no *tautologies or theorems* in the ordinary sense. *What might this mean for KS3$^{\Box}$?* While this is a rich topic all on its own, I will just gesture here at some interesting considerations.

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[6] Kripke (Saul Kripke, "Outline of a theory of truth," *Journal of Philosophy* 72, no. 19 (1975): 703) writes that, "what this means is that *if the interpretation of* T(x) *is extended by giving it a definite truth value for cases that were previously undefined, no truth value previously established changes or becomes undefined;* at most, certain previously undefined truth values become defined."

[7] Blamey () notes that by the introduction of modal operators "we have now introduced a crucial departure from the original models," in that certain orderings on the truth values no longer hold as they used to.

First, note that in Kleene propositional logic, there are no tautologies in the following sense: for any formula φ in Kleene propositional logic, there will be *at least one* row of the Kleene truth table for φ which evaluate to U; consequently, there are no formulae φ in Kleene propositional logic such that *all* rows of the Kleene truth table for φ evaluate to T, and *thus no classical tautologies*. The row which causes all the trouble, so to speak, is the row which assigns U to all propositional letters in φ, since the connectives in Kleene propositional logic always deliver U on this assignment. This made sense when we considered $\varphi = P \vee Q$: we reasoned that if P were undefined and Q were undefined, surely their disjunction must be undefined also. But we may feel differently when $\varphi = P \vee \neg P$ even though it shares the same form, since we typically take this φ—a statement of the law of the excluded middle—to be always true and provable in any satisfactory system. However, where P=U, Kleene's propositional logic evaluates φ as U altogether.

So, if we understand 'φ is a tautology' to require that φ takes the value T for all truth assignments of its propositional variables, then there are no tautologies in Kleene logic. However, there are different ways of specifying what is meant by a tautology which perhaps do better justice to the Kleene system. For example, while $P \vee \neg P$ is never always T, it *never evaluates to F*. Similarly, if we are inclined to think of U as intermediate in an ordering between T and F (say, it takes the value ½), then we may say that $P \vee \neg P$ *always evaluates to U or better.*

The logic KS3$^{\square}$ developed here may help elucidate the notion of a tautology in several ways. First, possible worlds are directly comparable to rows in a truth table, since each possible world consists of a set of propositions and a truth assignment on each proposition in that set. Consequently, it may seem that if φ evaluates to T at all rows of a truth table, then it should evaluate to T at all worlds. And the latter, of course, is just our notion of *necessity* in modal logic—assuming the appropriate accessibility relations are granted. An analysis of tautologies in terms of possible worlds seems almost *more* natural than truth tables, just because we are inclined to think of tautologies as having a special *modal* status: they are *true under any interpretation*. Often, we will explicitly state such claims in modal terms—say, that *there is no possible world at which φ (a tautology) is false*.

Now, just as there were no strict tautologies in the Kleene truth table, we may wonder about the corresponding concept of strict *necessity* in KS3$^{\square}$. Are there any formulae φ which, for any assignment of {T,U,F} to the propositional variables in φ evaluate to T? It may seem that this is the same as asking the φ for which $\vDash\Box\varphi$ are coextensional with those φ which are tautologies. Then, since we know that the row at which the Kleene truth table takes all U's it returns U also, it would follow that the evaluation would be otherwise identical if distributed on the set of possible worlds. For example, if we consider the classical tautology $\vDash P\vee\neg P$, we see that it is U at row U; correspondingly, at any world for which $V_w(P)=U$, then $V_w(P\vee\neg P)=U$ also. Thus, there is no strict necessity in $KS3^{\Box}$.

However, those sympathetic with this project will agree that this is by no means a damning problem, and to insist on *strict* necessity is to miss the point of the preceding considerations about the value of Kleene strong logic. Importantly, to my mind, such criticisms have no basis in *classical considerations*; that is, it is not open to a classically-minded detractor to object that *Kleene's system (or $KS3^{\Box}$) fails to give a desired classical answer*. For it may be proven simply (in much the same way as I have done here) that $KS3^{\Box}$ delivers classical results wherever the input is classical. To be more precise, for any formula φ with truth assignment in {T,F} to any propositional variables in φ, $V_w(\varphi)$ in $KS3^{\Box}$ = $V_w(\varphi)$ in classical logic. Thus, contra the classically-minded detractor, there is no φ for which $KS3^{\Box}$ fails to give the correct classical evaluation.

Indeed, it would be quite peculiar to object that there are no *strict* tautologies in Kleene logic, because such an objection cannot plausibly be motivated by a desire to preserve classical truth—this is aptly managed in Kleene logic. It occurs to me, lastly, that this final point may be brought out quite interestingly in the modal case, especially concerning *quantification* over disputed domains. In such cases, the more *permissive* disputant has an asymmetric advantage over his opponent.

There have been interesting discussions recently about quantificational difficulties facing metaphysical disputes regarding the *contents* of reality. For example, the dispute between nominalism and Platonism can be taken to be a dispute about the *size of the universe*. The Platonist believes the universe contains many things: for example, there is my sister and there is me, and also there *a property of siblinghood*. The nominalist, on the other hand, believes the universe is *small*: there is my sister and there is me, and that's it. The problem for the nominalist is this: how does he mount his objection to the Platonist? This may seem easy; he just says that "there is no such thing as a property of siblinghood." But this sentence, any hawk-like logician will observe, contains a quantificational operator; and, here's the rub, the operator supposedly ranges over *everything*—over the entire contents of reality. So, the nominalist is caught in Plato's beard: he wishes to assert that there is no object {siblinghood}, but in order to make this assertion he needs to refer to this very object. This would not be a problem if we were speaking loosely—as I am here, and we do usually—but since the nominalist is intending to speak *strictly* and *in terms of* the fundamental constituents of reality *only*, he cannot countenance reference to items outside his universe of discourse; indeed, he cannot understand them.

It occurs to me that a *nominalistically* classically minded-detractor, N, of my KS3$^{\square}$ may be prey to similar problems. My earlier detractor lamented that in KS3$^{\square}$ there are no strict tautologies, because he wanted $P \lor \neg P$ to evaluate to T on *all* assignments. Unlike this detractor, my new detractor N is upset because, to his mind, $P \lor \neg P$ should evaluate to T on *only those assignments found in the classical truth table*. Now, in KS3$^{\square}$ there is at least one assignment which evaluates to T and which is not found in the classical truth table—specifically at the assignment {T,U}. Now, how might my detractor make his claim? It may be observed that he strictures preclude him from making his objection: if he wishes to speak strictly about what objects there are in the universe (and this is the goal of his objection, after all), then presumably he may make reference only to what exists. But what exists—from his perspective—are only those worlds and constituents in the classical domain, which is a proper subset of the Kleene worlds. Consequently, the world *w* expressed by $\{P \lor \neg P \mid T,U\}$ is outside of the domain and the language of my detractor; and so the objection cannot be articulated.

Whether such arguments are ultimately successful or not, it is clear that the defender of Kleene logic is in a strong position, not least because Kleene logic is able to deliver *all* the classical results—and then some.

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